Simulating Events of Unknown Probabilities via Reverse Time Martingales

K. Latuszynski*   I. Kosmidis

O. Papaspiliopoulos   G.O. Roberts

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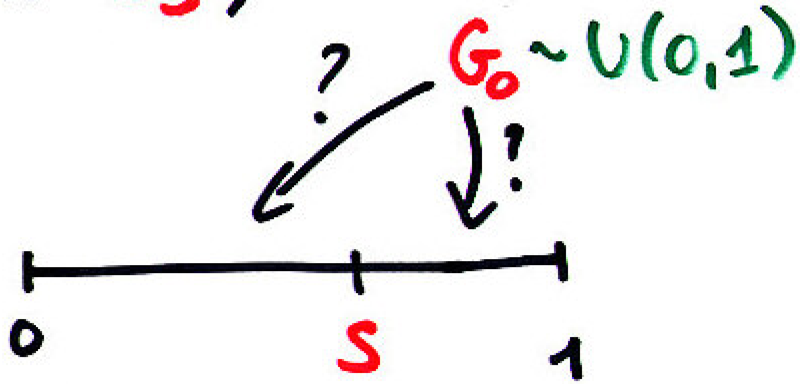

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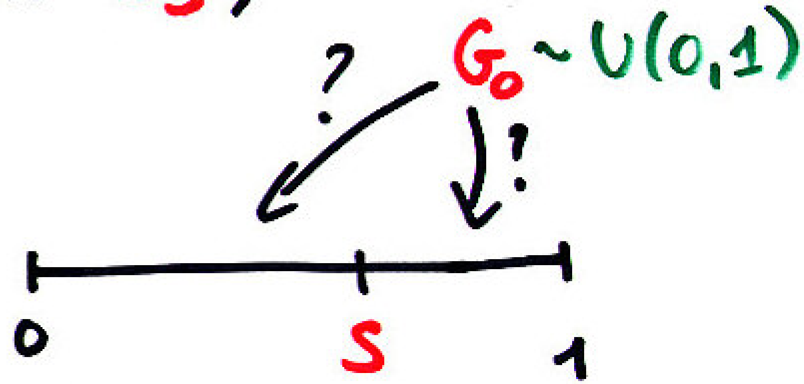

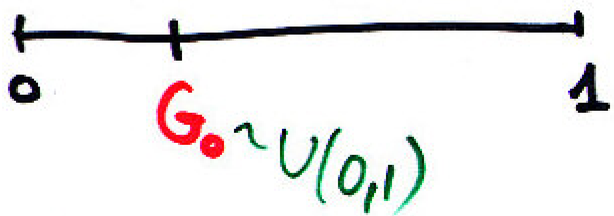

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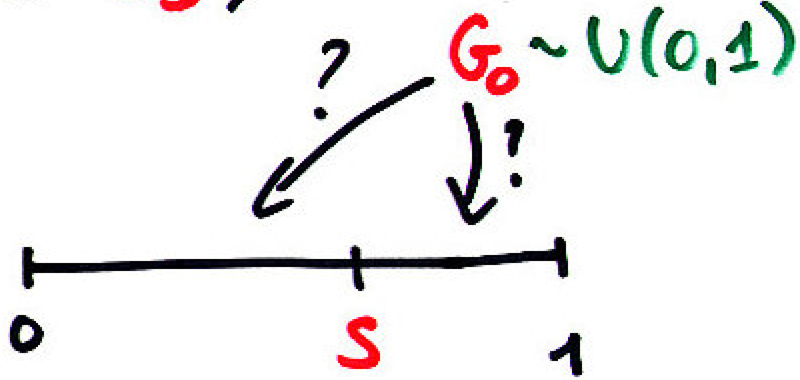

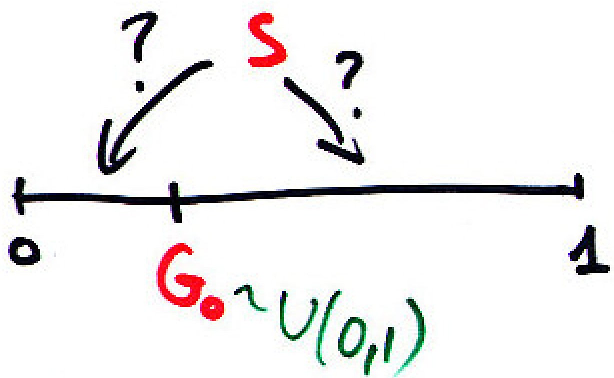

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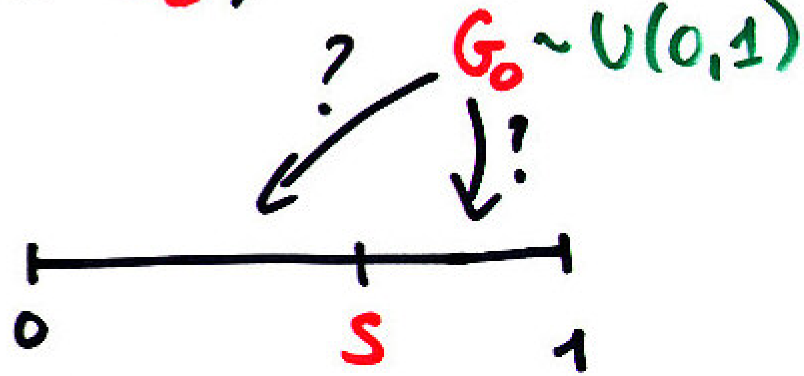

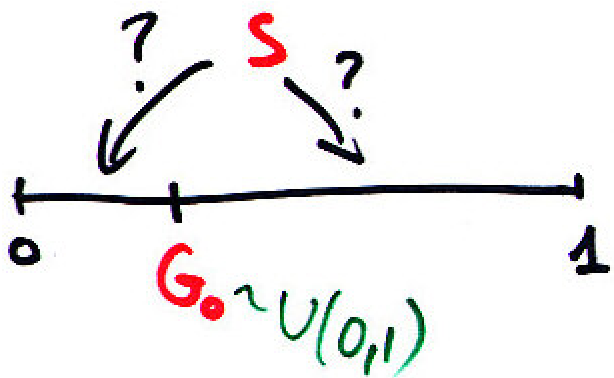

- We assume that s is uniquely determined but not known explicitly

- Motivation: often an element of more complex simulation procedures.

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- MCMC inference for diffusion parameter
- rejection sampling
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how to simulate an $f(p)$-coin given an iid sequence of p-coins

p - UNKNOWN

f KNOWN ex $f(p)=2p$

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4. if $G_0 \geq u_n$ set $C_S := 0$
5. if $l_n < G_0 < u_n$ set $n = n+1$ and GOTO 2.
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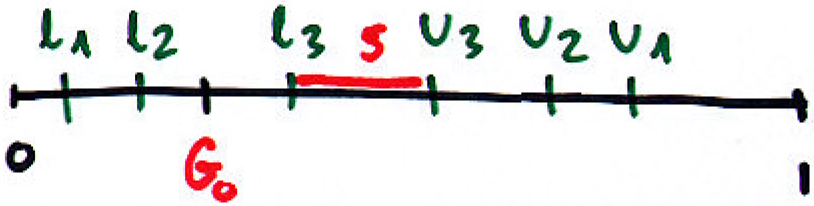

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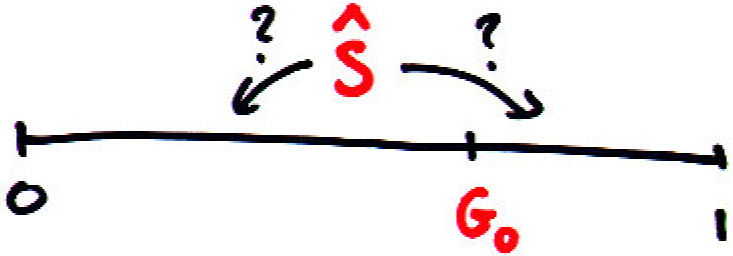
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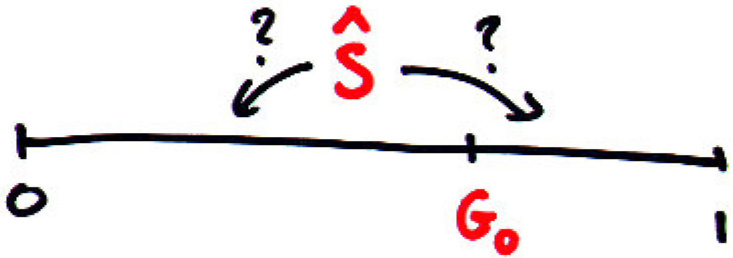
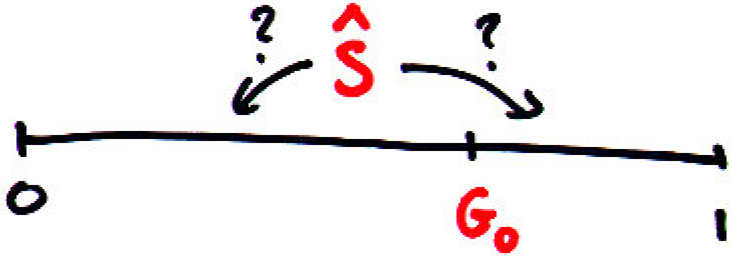


Assume $\hat{S}$ is an unbiased estimator of S.



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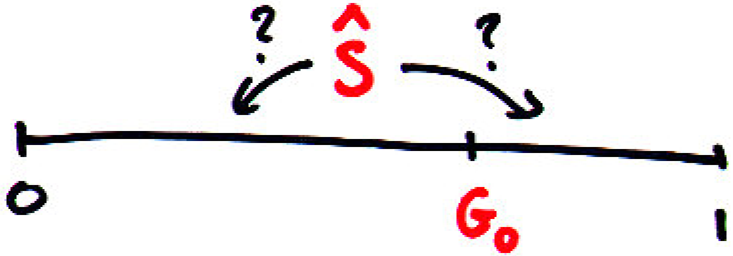

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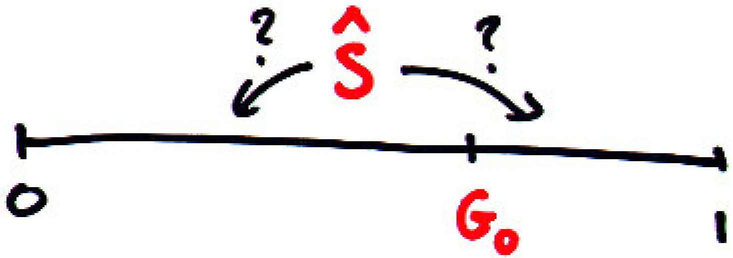

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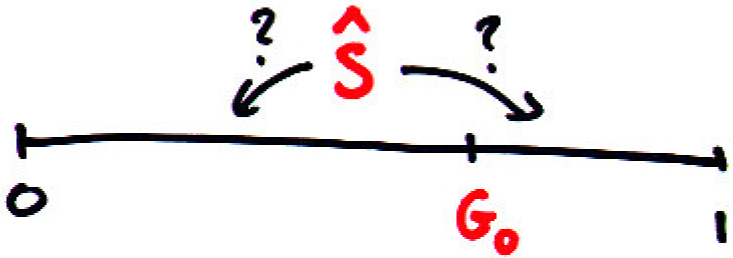

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- $E L_n = l_n \nearrow S$ and $E U_n = u_n \searrow S$

L_n and U_n - unbiased estimators of l_n and u_n (6)

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Algorithm 3

1. $G_0 \sim U(0,1)$; $n = 1$
2. obtain L_n and U_n given L_{n-1}, U_{n-1}
3. if $G_0 \le L_n$ ...

   $G_0 \ge U_n$ ...

   $L_n < G_0 < U_n$ ...

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Lemma

Algorithm 3 outputs a valid s-coin

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$\mathcal{F}_0 = \{\phi, \Omega\}$ $\mathcal{F}_n = \sigma(L_n, U_n)$

$\mathcal{F}_{k,n} = \sigma(\mathcal{F}_k, \mathcal{F}_{k+1}, \dots \mathcal{F}_n)$

given $\mathcal{F}_{0,n-1}$

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(**) L_n - reverse time supermartingale

U_n - reverse time submartingale

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$$E(L_{n-1} \mid \mathcal{F}_{n,\infty}) = E(L_{n-1} \mid \mathcal{F}_n) \leq L_n$$

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3. compute $L_n^* = E(L_{n-1} | \mathcal{F}_n)$ and $U_n^* = E(U_{n-1} | \mathcal{F}_n)$

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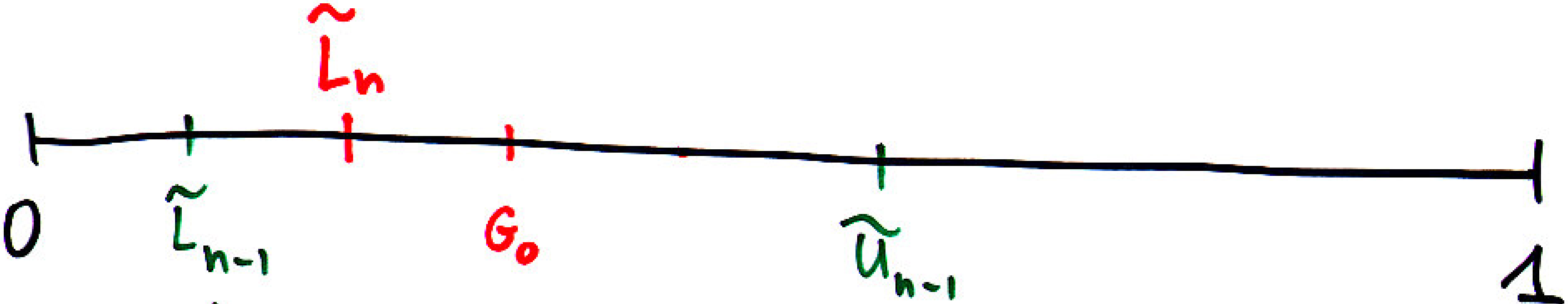
....


0
$\tilde{L}_{n-1}$
G_0
$\tilde{U}_{n-1}$
1



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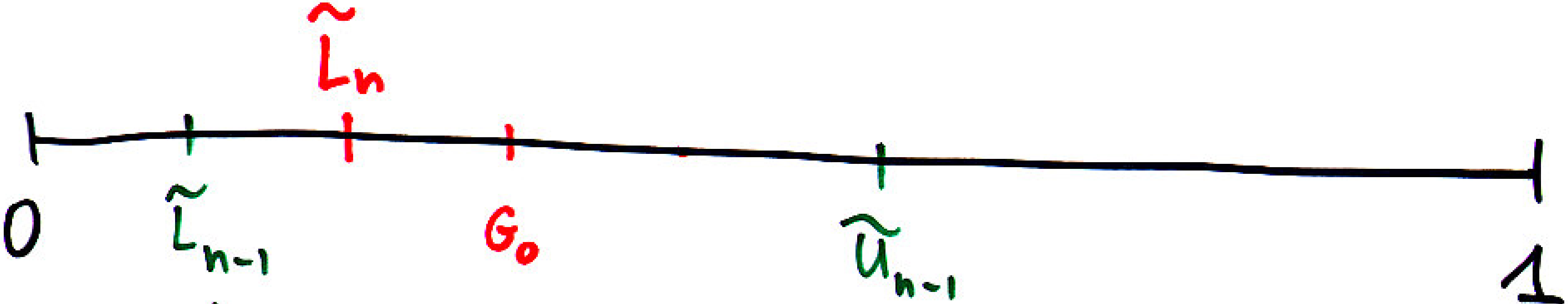

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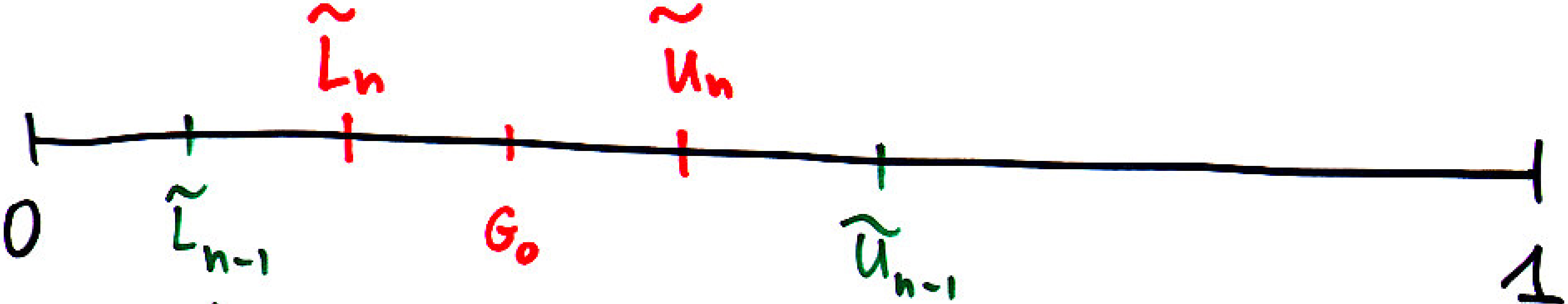

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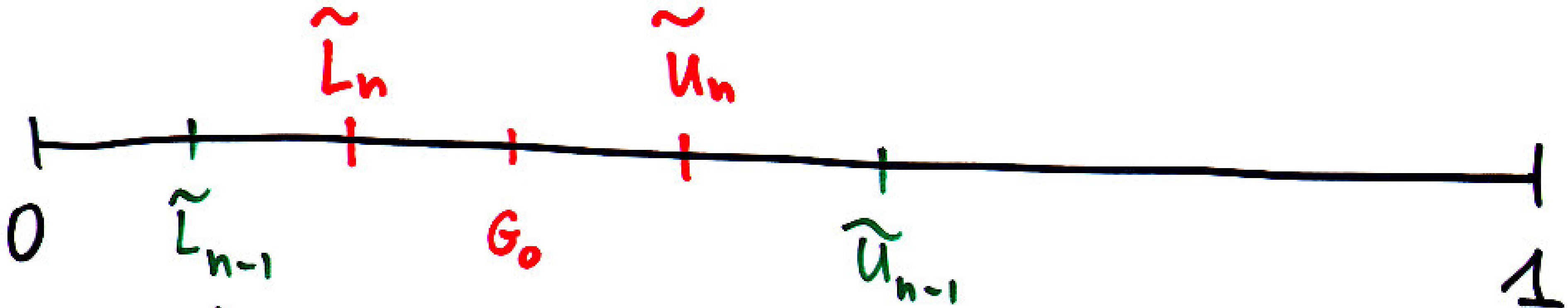

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This construction preserves expectation and $\tilde{L}_n, \tilde{U}_n$ are monotone.

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__Thm.__ Algorithm 4 outputs a valid s-coin.

The Bernoulli Factory Problem

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input: $X_1, X_2, \ldots$ sequence of p-coins (p is UNKNOWN)
$p \in (0, 1/2)$

output: Y a single $2p$-coin

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$$g_n(p) = \sum_{k=0}^{n} \binom{n}{k} a(n,k) p^k (1-p)^{n-k} \qquad h_n(p) = \sum_{k=0}^{n} \binom{n}{k} b(n,k) p^k (1-p)^{n-k}$$

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(i) $0 \leq a(n,k) \leq b(n,k) \leq 1$

(ii) $\lim\limits_{n \to \infty} g_n(p) = f(p) = \lim\limits_{n \to \infty} h_n(p)$

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input: $X_1, X_2, \dots$ sequence of p-coins (p is UNKNOWN)
$p \in (0, 1/2)$

output: Y a single $2p$-coin $\to$ IMPOSSIBLE!

Y a single $f(p) = \min\{2p, 1-2\varepsilon\}$-coin

Proposition: An algorithm that simulates f on $\mathcal{P} \subseteq (0,1)$ exists if and only if for all $n \geq 1$ there exist polynomials $g_n(p)$ and $h_n(p)$ of the form

$$g_n(p) = \sum_{k=0}^{n} \binom{n}{k} a(n,k) p^k (1-p)^{n-k} \qquad h_n(p) = \sum_{k=0}^{n} \binom{n}{k} b(n,k) p^k (1-p)^{n-k}$$

(i) $0 \leq a(n,k) \leq b(n,k) \leq 1$

(ii) $\lim_{n\to\infty} g_n(p) = f(p) = \lim_{n\to\infty} h_n(p)$

(iii) for all $m < n$ coefficients satisfy

$$a(n,k) \geq \sum_{i=0}^{k} \frac{\binom{n-m}{k-i}\binom{m}{i}}{\binom{n}{k}} a(m,i) \qquad b(n,k) \leq \sum_{i=0}^{k} \frac{\binom{n-m}{k-i}\binom{m}{i}}{\binom{n}{k}} b(m,i)$$

polynomials $\Rightarrow$ algorithm

$X_1, X_2, \ldots$ p-coins if $\sum_{i=1}^{n} X_i = k$, let $L_n = a(n,k)$

$U_n = b(n,k)$

The rest of the proof:

check that assumptions for L_n and U_n

in ALgorithm 4 hold!